Exercises - Univariate resultants

M2 MPA - Computational Algebraic Geometry

November 9, 2020

Exercice 1 (Sylvester resultant)**.** The goal of this exercise is to review some properties of the classical Sylvester resultant.
Let A be a commutative ring and consider the polynomials

$$\begin{cases} f(x) &= a_0x^m + a_1x^{m-1} + \cdots + a_m \\ g(x) &= b_0x^n + b_1x^{n-1} + \cdots + b_n \end{cases} \tag{1}$$

of positive degree with f and g in $A[x]$. The Sylvester matrix of f and g is defined as

$$S_{m,n}(f,g) = \begin{pmatrix} a_0 & 0 & \cdots & 0 & b_0 & 0 & 0 \\ a_1 & a_0 & & \vdots & b_1 & \ddots & 0 \\ \vdots & & \ddots & 0 & \vdots & & b_0 \\ a_m & & & a_0 & b_{n-1} & & b_1 \\ 0 & a_m & & a_1 & b_n & & \vdots \\ \vdots & & \ddots & \vdots & 0 & \ddots & b_{n-1} \\ 0 & \cdots & 0 & a_m & 0 & 0 & b_n \end{pmatrix}.$$

This is a square matrix of size $(m+n)$; its determinant is the so-called Sylvester resultant of $f(x)$ and $g(x)$, denoted $\mathrm{Res}_{m,n}(f,g)$, or simply $\mathrm{Res}(f,g)$ if there is no confusion. Observe that by definition, we have the equality

$$S_{m,n}(f,g)^T \begin{pmatrix} 1 \\ x \\ \vdots \\ x^{m+n-2} \\ x^{m+n-1} \end{pmatrix} = \begin{pmatrix} f \\ xf \\ \vdots \\ x^{n-1}f \\ g \\ xg \\ \vdots \\ x^{m-1}g \end{pmatrix}$$

in $A[x]$, where $(-)^T$ stands for the transpose matrix.
The polynomials f and g define a map of free $A[x]$-modules

$$A[x] \oplus A[x] \to A[x] : (u,v) \mapsto uf + vg$$

that induces another map of free A-modules by restriction ($A[x]_{<d}$ denotes the set of polynomials of degree $< d$):

$$\phi : A[x]_{<n} \times A[x]_{<m} \to A[x]_{<m+n} : (u,v) \mapsto uf + vg.$$

The Sylvester matrix of f and g is nothing but the matrix of ϕ in canonical basis. In particular, if A is a domain then ϕ is injective if and only if $\mathrm{Res}(f,g) \neq 0$.

1. Assume that A is a domain and let $k = \mathrm{Frac}(A)$ be its fraction field. Let f and g be two polynomials in $A[x]$ defined by (1) and such that $a_0 \neq 0$. Then, show that $\mathrm{Res}_{m,n}(f,g) \neq 0$ if and only if $f(x)$ and $g(x)$ are relatively prime polynomials in $k[x]$. In particular $\mathrm{Res}_{m,n}(f,g) \neq 0$ if and only if f and g has no common root in the algebraic closure of k.

2. Assume that $A = k$ is a field and that $(a_0, b_0) \neq (0,0)$. Show first that

$$\operatorname{corank} S_{m,n}(f,g) = m + n - \operatorname{rank} S_{m,n}(f,g) = \deg \gcd(f,g).$$

   Then, assuming that $\gcd(f,g) = \prod_{i=1}^{r}(x-\alpha_i)^{m_i}$, $\alpha_i \neq \alpha_j$ in some extension $\bar{k}$ of k, show that a basis of the cokernel of $S_{m,n}(f,g)$ is given by the columns of the matrix V which is built by putting side by side the generalized (or confluent) Vandermonde block matrices

$$V_{m+n-1}(\alpha_1; m_1), V_{m+n-1}(\alpha_2; m_2), \ldots, V_{m+n-1}(\alpha_r; m_r),$$

   where

$$V_d(\alpha;k) = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ \alpha & 1 & \cdots & 0 \\ \alpha^2 & 2\alpha & \cdots & 0 \\ \alpha^3 & 3\alpha^2 & \ddots & \vdots \\ \vdots & \vdots & \cdots & \frac{(d-1)!}{(d-k-1)!}\alpha^{d-k} \\ \alpha^d & d\alpha^{d-1} & \cdots & \frac{d!}{(d-k)!}\alpha^{d-k+1} \end{pmatrix}.$$

   (Hint: use the known fact that the determinant of a generalized Vandermonde square matrix – take $d = \sum_i m_i$ above – is equal to $\prod_{i<j}(x_i - x_j)^{m_i m_j}$).

3. Let Δ_0 be the top square block of V of maximal size $\sum_{i=1}^{r} m_i = \deg \gcd(f,g)$ and define Δ_1 similarly with a shift down by one row. Show that the generalized eigenvalues of the pencil (Δ_1, Δ_0) are $\alpha_1, \ldots, \alpha_r$ with multiplicity $m_1, \ldots, m_r$ respectively.

Exercice 2 (Geometry of the Sylvester resultant)**.** Let k be an algebraically closed field. Given two positive integers m, n, we consider couples of homogeneous polynomials

$$f(x,y) = a_0 x^m + a_1 x^{m-1} y + \cdots + a_m y^m$$
$$g(x,y) = b_0 x^n + b_1 x^{n-1} y + \cdots + b_n y^n$$

in the variables x, y with coefficients in k. Up to multiplication by nonzero constants in k, these couples are in bijection with a product of two projective spaces, namely $\mathbb{P}^m \times \mathbb{P}^n$. Thus, f and g define an incidence variety $W = V(f,g)$ in $\mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^n$. If π denotes the canonical projection

$$\pi : \mathbb{P}^1 \times \mathbb{P}^m \times \mathbb{P}^n \to \mathbb{P}^m \times \mathbb{P}^n$$

then the image of W via π is the resultant variety $V(\mathrm{Res}(f,g))$.

1. Let (p,q) by a point in $\mathbb{P}^m \times \mathbb{P}^n$. Justify that the degree of the fiber of π at (p,q), counting multiplicities, is equal to the corank of the corresponding Sylvester matrix, i.e.

$$\operatorname{corank} S_{m,n}(p,q) = \deg(\pi^{-1}(p,q)).$$

2. Prove that the multiplicity of the point (p, q) on the resultant variety, equivalently the order (or valuation) of the resultant at the point (p, q), is equal to the degree of the fiber of π above (p, q), counting multiplicities.

Exercice 3 (Hybrid-Bézout matrices)**.** The goal of this exercise is to introduce matrices of lower size than the Sylvester matrix but with similar properties.
Let $f(x), g(x)$ be two polynomials of degree m, n respectively, as defined in (1), and let $\alpha := (\alpha_1, \alpha_2)$ be any couple of non-negative integers such $|\alpha| := \alpha_1 + \alpha_2 \leq \min\{m, n\} - 1$; for simplicity, we assume that $m \leq n$. One can decompose f and g as

$$f = x^{\alpha_1+1} h_{1,1} + x^{\alpha_2+1} h_{1,2},$$
$$g = y^{\alpha_1+1} h_{2,1} + y^{\alpha_2+1} h_{2,2},$$

where $h_{i,j}(x, y)$ are homogeneous polynomials of degree $d_i - \alpha_j - 1$, and define the polynomial

$$\mathrm{syl}_\alpha(f, g) := \det \begin{pmatrix} h_{1,1} & h_{1,2} \\ h_{2,1} & h_{2,2} \end{pmatrix}.$$

This latter polynomial is called a *Sylvester form* of f and g. It is of degree $m+n-2-|\alpha|$ with respect to x, y and of degree 2 with respect to the coefficients of f and g.
Now, for all $k = 0, \ldots, m-1$ define the matrix

$$H_k = \left(\begin{array}{cccc|ccc|ccc} a_0 & 0 & \cdots & 0 & b_0 & 0 & 0 & & & \\ a_1 & a_0 & & \vdots & b_1 & \ddots & 0 & \vdots & \vdots & \vdots \\ \vdots & & \ddots & 0 & \vdots & & b_0 & & & \\ a_m & & & a_0 & b_{n-1} & & b_1 & \mathrm{syl}_\alpha(f,g) & \cdots & \mathrm{syl}_\alpha(f,g) \\ 0 & a_m & & a_1 & b_n & & \vdots & & & \\ \vdots & & \ddots & \vdots & 0 & \ddots & b_{n-1} & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & a_m & 0 & 0 & b_n & & & \end{array}\right)$$

such that: the rows of H_k are indexed by the monomial basis $1, x, \ldots, x^{m+n-k-1}$, the two blocks from the left are Sylvester-like blocks with $n-k-1$ columns depending on f and $m-k-1$ columns depending on g, and the block on the right side is built by columns with the coefficients of the $k+1$ Sylvester forms $\mathrm{syl}_\alpha(f, g)$ with $|\alpha| = k$. The matrices H_k are commonly called hybrid-Bézout matrices.

Show that the determinant of $H_k(f, g)$ is equal to the resultant of f and g (up to a nonzero multiplicative constant).